

# On the well-posedness of SPDEs with singular drift in divergence form

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January 28, 2017

## Abstract

We prove existence and uniqueness of strong solutions for a class of second-order stochastic PDEs with multiplicative Wiener noise and drift of the form  $\operatorname{div} \gamma(\nabla \cdot)$ , where  $\gamma$  is a maximal monotone graph in  $\mathbb{R}^n \times \mathbb{R}^n$  obtained as the subdifferential of a convex function satisfying very mild assumptions on its behavior at infinity. The well-posedness result complements the corresponding one in our recent work arXiv:1612.08260 where, under the additional assumption that  $\gamma$  is single-valued, a solution with better integrability and regularity properties is constructed. The proof given here, however, is self-contained.

## 1 Introduction and main result

Let us consider the stochastic partial differential equation

$$du(t) - \operatorname{div} \gamma(\nabla u(t)) dt \ni B(t, u(t)) dW(t), \quad u(0) = u_0, \quad (1)$$

posed on  $L^2(D)$ , with  $D$  a bounded domain of  $\mathbb{R}^n$  with smooth boundary. The following assumptions will be in force: (a)  $\gamma$  is the subdifferential of a lower semicontinuous convex function  $k : \mathbb{R}^n \rightarrow \mathbb{R}_+$  with  $k(0) = 0$  and such that

$$\lim_{|x| \rightarrow \infty} \frac{k(x)}{|x|} = +\infty, \quad \limsup_{|x| \rightarrow \infty} \frac{k(-x)}{k(x)} < +\infty$$

(in particular,  $\gamma$  is a maximal monotone graph in  $\mathbb{R}^n \times \mathbb{R}^n$  whose domain coincides with  $\mathbb{R}^n$ ); (b)  $W$  is a cylindrical Wiener process on a separable Hilbert space  $H$ , supported by a filtered probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$  satisfying the “usual conditions”; (c)  $B$  is a map from  $\Omega \times [0, T] \times L^2(D)$  to  $\mathcal{L}^2(H, L^2(D))$ , the space of Hilbert-Schmidt operators from  $H$  to  $L^2(D)$ , that is Lipschitz-continuous and has linear growth with respect to its third argument, uniformly with respect to the other two, and is such that  $B(\cdot, \cdot, a)$  is measurable and adapted for all  $a \in L^2(D)$ .

Under the additional assumption that  $\gamma$  is a (single-valued) continuous function, we proved in [7] that (1) admits a strong solution  $u$ , which is unique within a set of processes satisfying mild integrability conditions. The solution of [7] is constructed pathwise, i.e. for each  $\omega \in \Omega$ , so that, as is natural to expect, measurability problems arise with respect to the usual  $\sigma$ -algebras on  $\Omega \times [0, T]$  used in the theory of stochastic processes. Precisely because of such an issue we needed to assume  $\gamma$  to be single-valued.

The purpose of this note is to provide an alternative approach to establish the well-posedness of (1) that, avoiding pathwise constructions, is simpler than that of [7] and does not need any extra assumption on  $\gamma$ . The price to pay is that the solution we obtain here is less regular than that of [7]. We also refer to [9] for a related result obtained by analogous methods.

Let us define the concept of solution to (1) we shall be working with.

**Definition 1.1.** *Let  $u_0$  be an  $L^2(D)$ -valued  $\mathcal{F}_0$ -measurable random variable. A strong solution to equation (1) is a couple  $(u, \eta)$  satisfying the following properties:*

(i)  *$u$  is a measurable and adapted  $L^2(D)$ -valued process such that*

$$u \in L^1(0, T; W_0^{1,1}(D)) \quad \text{and} \quad B(\cdot, u) \in L^2(0, T; \mathcal{L}^2(U, L^2(D))) \quad \mathbb{P}\text{-a.s.};$$

(ii)  *$\eta$  is a measurable and adapted  $L^1(D)^n$ -valued process such that*

$$\eta \in L^1(0, T; L^1(D)^n) \quad \mathbb{P}\text{-a.s.}, \quad \eta \in \gamma(\nabla u) \quad \text{a.e. in } \Omega \times (0, T) \times D;$$

(iii) *one has, as an equality in  $L^2(D)$ ,*

$$u(t) - \int_0^t \operatorname{div} \eta(s) ds = u_0 + \int_0^t B(s, u(s)) dW(s) \quad \mathbb{P}\text{-a.s.} \quad \forall t \in [0, T]. \quad (2)$$

Note that (2) has to be intended in the sense of distributions. In particular, since  $\eta \in L^1(D)^n$ , the integrand in the second term of (2) does not, in general, take values in  $L^2(D)$ . However, the conditions on  $B$  imply that the stochastic integral in (2) is an  $L^2(D)$ -valued local martingale, hence the term involving the divergence of  $\eta$  turns out to be  $L^2(D)$ -valued by comparison.

We can now state our main result. Here and in the following  $k^* : \mathbb{R}^n \rightarrow \mathbb{R}_+$  is the convex conjugate of  $k$ , defined as  $k^*(y) := \sup_{x \in \mathbb{R}^n} (x \cdot y - k(x))$ .

**Theorem 1.2.** *Let  $u_0 \in L^2(\Omega, \mathcal{F}_0; L^2(D))$ . Then equation (1) admits a unique strong solution  $(u, \eta)$  such that*

$$\begin{aligned} & \sup_{t \leq T} \mathbb{E} \|u(t)\|_{L^2(D)}^2 + \mathbb{E} \int_0^T \|u(t)\|_{W_0^{1,1}(D)} dt < \infty, \\ & \mathbb{E} \int_0^T \|\eta(t)\|_{L^1(D)^n} dt < \infty, \\ & \mathbb{E} \int_0^T (\|k(\nabla u(t))\|_{L^1(D)} + \|k^*(\eta(t))\|_{L^1(D)}) dt < \infty. \end{aligned}$$

Moreover, the solution map  $u_0 \mapsto u$  is Lipschitz-continuous from  $L^2(\Omega; L^2(D))$  to  $L^\infty(0, T; L^2(\Omega; L^2(D)))$ , and  $u$  is weakly continuous as a function on  $[0, T]$  with values in  $L^2(\Omega; L^2(D))$ .

Under the extra assumption of  $\gamma$  being single-valued, the solution obtained in [7] is more regular in the sense that  $\mathbb{E} \sup_{t \leq T} \|u(t)\|_{L^2(D)}^2$  is finite, the solution map is Lipschitz-continuous from  $L^2(\Omega; L^2(D))$  to  $L^2(\Omega; L^\infty(0, T; L^2(D)))$ , and  $u(\omega, \cdot)$  is weakly continuous as a function on  $[0, T]$  with values in  $L^2(D)$  for  $\mathbb{P}$ -a.a.  $\omega \in \Omega$ .

**Acknowledgements.** The authors are partially supported by The Royal Society through its International Exchange Scheme. Parts of this chapter were written while the first-named author was visiting the Interdisziplinäres Zentrum für Komplexe Systeme at the University of Bonn, hosted by Prof. S. Albeverio.

## 2 Well-posedness of an auxiliary equation

The goal of this section is to prove well-posedness of a version of (1) with additive noise. Namely, we consider the initial value problem

$$du(t) - \operatorname{div} \gamma(\nabla u(t)) dt \ni G(t) dW(t), \quad u(0) = u_0, \quad (3)$$

where  $G \in L^2(\Omega \times [0, T]; \mathcal{L}^2(H, L^2(D)))$  is a measurable and adapted process.

**Proposition 2.1.** *Equation (3) admits a unique strong solution  $(u, \eta)$  satisfying the same integrability and weak continuity conditions of Theorem 1.2.*

We introduce the regularized equation

$$du_\lambda(t) - \operatorname{div} \gamma_\lambda(\nabla u_\lambda(t)) dt - \lambda \Delta u_\lambda(t) dt = G(t) dW(t), \quad u_\lambda(0) = u_0,$$

where  $\gamma_\lambda : \mathbb{R}^n \rightarrow \mathbb{R}^n$ ,  $\gamma_\lambda := \frac{1}{\lambda}(I - (I + \lambda\gamma)^{-1})$ , for any  $\lambda > 0$ , is the Yosida approximation of  $\gamma$ , and  $\Delta : H_0^1(D) \rightarrow H^{-1}(D)$  is the (variational) Dirichlet Laplacian. Since  $\gamma_\lambda$  is monotone and Lipschitz-continuous, the classical variational

approach (see [4, 8] as well as [5]) yields the existence of a unique predictable process  $u_\lambda$  with values in  $H_0^1(D)$  such that

$$\mathbb{E}\|u_\lambda\|_{C([0,T];L^2(D))}^2 + \mathbb{E} \int_0^T \|u_\lambda(t)\|_{H_0^1(D)}^2 dt < \infty$$

and

$$u_\lambda(t) - \int_0^t \operatorname{div} \gamma_\lambda(\nabla u_\lambda(s)) ds - \lambda \int_0^t \Delta u_\lambda(s) ds = u_0 + \int_0^t G(s) dW(s) \quad (4)$$

$\mathbb{P}$ -a.s. in  $H^{-1}(D)$  for all  $t \in [0, T]$ .

We are now going to prove a priori estimates and weak compactness in suitable topologies for  $u_\lambda$  and related processes. These will allow us to pass to the limit as  $\lambda \rightarrow 0$  in (4).

For notational parsimony, we shall often write, for any  $p \geq 0$ ,  $L_\omega^p$ ,  $L_t^p$ , and  $L_x^p$  in place of  $L^p(\Omega)$ ,  $L^p(0, T)$ , and  $L^p(D)$ , respectively, and  $C_t$  to denote  $C([0, T])$ . Other similar abbreviations are self-explanatory. The  $L^2(D)$ -norm will be denoted simply by  $\|\cdot\|$ . If a function  $f : D \rightarrow \mathbb{R}^n$  is such that each component  $f^j$ ,  $j = 1, \dots, n$ , belongs to  $L^p(D)$ , we shall just write  $f \in L^p(D)$  rather than  $f \in L^p(D)^n$ . The notation  $a \lesssim b$  means that  $a \leq Nb$  for a positive constant  $N$ .

**Lemma 2.2.** *There exists a constant  $N$  such that*

$$\begin{aligned} & \|u_\lambda\|_{L_\omega^2 C_t L_x^2} + \lambda^{1/2} \|\nabla u_\lambda\|_{L_{t,\omega,x}^2} + \|\gamma_\lambda(\nabla u_\lambda) \cdot \nabla u_\lambda\|_{L_{t,\omega,x}^1} \\ & < N(\|u_0\|_{L_{\omega,x}^2} + \|G\|_{L_{t,\omega}^2 \mathcal{L}^2(H, L_x^2)}). \end{aligned}$$

*Proof.* Itô's formula for the square of the norm in  $L_x^2$  yields

$$\begin{aligned} & \|u_\lambda(t)\|^2 + 2 \int_0^t \int_D \gamma(\nabla u_\lambda(s)) \cdot \nabla u_\lambda(s) dx ds + 2\lambda \int_0^t \|\nabla u_\lambda(s)\|^2 ds \\ & = \|u_0\|^2 + 2 \int_0^t u_\lambda(s) G(s) dW(s) + \int_0^t \|G(s)\|_{\mathcal{L}^2(H, L_x^2)}^2 ds, \end{aligned}$$

hence, taking the supremum in time and expectation,

$$\begin{aligned} & \mathbb{E}\|u_\lambda\|_{C_t L_x^2}^2 + \mathbb{E} \int_0^T \int_D \gamma_\lambda(\nabla u_\lambda(s)) \cdot \nabla u_\lambda(s) dx ds + \lambda \mathbb{E}\|\nabla u_\lambda\|_{L_{t,x}^2}^2 \\ & \lesssim \mathbb{E}\|u_0\|^2 + \mathbb{E}\|G\|_{L_t^2 \mathcal{L}^2(H, L_x^2)}^2 + \mathbb{E} \sup_{t \in [0, T]} \left| \int_0^t u_\lambda(s) G(s) dW(s) \right|, \end{aligned}$$

where, by Davis' inequality (see, e.g., [6]), the ideal property of Hilbert-Schmidt operators (see, e.g., [1, p. V.52]), and the elementary inequality  $ab \leq \varepsilon a^2 + b^2/\varepsilon$

$\forall a, b \geq 0, \varepsilon > 0,$

$$\begin{aligned} \mathbb{E} \sup_{t \in [0, T]} \left| \int_0^t u_\lambda(s) G(s) dW(s) \right| &\lesssim \mathbb{E} \left( \int_0^T \|u_\lambda(s) G(s)\|_{\mathcal{L}^2(H, \mathbb{R})}^2 ds \right)^{1/2} \\ &\leq \varepsilon \mathbb{E} \|u_\lambda\|_{C_t L_x^2}^2 + N(\varepsilon) \mathbb{E} \int_0^T \|G(s)\|_{\mathcal{L}^2(H, L_x^2)}^2 ds \end{aligned}$$

for any  $\varepsilon > 0$ . To conclude it suffices to choose  $\varepsilon$  small enough.  $\square$

**Lemma 2.3.** *The families  $(\nabla u_\lambda)$  and  $(\gamma_\lambda(\nabla u_\lambda))$  are relatively weakly compact in  $L^1(\Omega \times (0, T) \times D)$ .*

*Proof.* Recall that, for any  $y, r \in \mathbb{R}^n$ , ones has  $k(y) + k^*(r) = r \cdot y$  if and only if  $r \in \partial k(y) = \gamma(y)$ . Therefore, since

$$\gamma_\lambda(x) \in \partial k((I + \lambda\gamma)^{-1}x) = \gamma((I + \lambda\gamma)^{-1}x) \quad \forall x \in \mathbb{R}^n,$$

we deduce by the definition of  $\gamma_\lambda$  that

$$\begin{aligned} k((I + \lambda\gamma)^{-1}x) + k^*(\gamma_\lambda(x)) &= \gamma_\lambda(x) \cdot (I + \lambda\gamma)^{-1}x \\ &= \gamma_\lambda(x) \cdot x - \lambda |\gamma_\lambda(x)|^2 \leq \gamma_\lambda(x) \cdot x \quad \forall x \in \mathbb{R}^n. \end{aligned} \tag{5}$$

(See, e.g., [3] for all necessary facts from convex analysis used in this note.) Hence, taking Lemma 2.2 into account, there exists a constant  $N > 0$ , independent of  $\lambda$ , such that

$$\mathbb{E} \int_0^T \int_D k^*(\gamma_\lambda(\nabla u_\lambda)) \leq \mathbb{E} \int_0^T \int_D \gamma_\lambda(\nabla u_\lambda) \cdot \nabla u_\lambda < N.$$

The assumptions on  $k$  imply that its convex conjugate  $k^*$  is also convex, lower semicontinuous and such that  $\lim_{|y| \rightarrow \infty} k^*(y)/|y| = +\infty$ . Therefore a simple modification of the criterion by de la Vallée Poussin implies that  $(\gamma_\lambda(\nabla u_\lambda))$  is uniformly integrable on  $\Omega \times (0, T) \times D$ , hence that it is relatively weakly compact in  $L^1_{t, \omega, x}$  by the Dunford-Pettis theorem. A completely analogous argument shows that

$$\mathbb{E} \int_0^T \int_D k((I + \lambda\gamma)^{-1} \nabla u_\lambda) \leq \mathbb{E} \int_0^T \int_D \gamma_\lambda(\nabla u_\lambda) \cdot \nabla u_\lambda < N,$$

hence that  $(I + \lambda\gamma)^{-1} \nabla u_\lambda$  is relatively weakly compact in  $L^1_{t, \omega, x}$ . Moreover, since  $\nabla u_\lambda = (I + \lambda\gamma)^{-1} \nabla u_\lambda + \lambda \gamma_\lambda(\nabla u_\lambda)$ , it also follows that  $(\nabla u_\lambda)$  is relatively weakly compact in  $L^1_{t, \omega, x}$ .  $\square$

Thanks to Lemmata 2.2 and 2.3, there exists a subsequence of  $\lambda$ , denoted by the same symbol, and processes  $u \in L_t^\infty L_{\omega,x}^2 \cap L_{t,\omega}^1 W_0^{1,1}$  and  $\eta \in L_{t,\omega,x}^1$  such that

$$\begin{aligned} u_\lambda &\longrightarrow u && \text{weakly* in } L_t^\infty L_{\omega,x}^2, \\ u_\lambda &\longrightarrow u && \text{weakly in } L_{t,\omega}^1 W_0^{1,1}, \\ \gamma_\lambda(\nabla u_\lambda) &\longrightarrow \eta && \text{weakly in } L_{t,\omega,x}^1, \\ \lambda u_\lambda &\longrightarrow 0 && \text{weakly in } L_{t,\omega}^2 H_0^1. \end{aligned}$$

as  $\lambda \rightarrow 0$ . Let  $t \in [0, T]$  be arbitrary but fixed. The fourth convergence above implies

$$\lambda \int_0^t \Delta u_\lambda(s) ds \longrightarrow 0 \quad \text{in } L_\omega^2 H^{-1},$$

while the third yields, for any  $\varphi \in L_\omega^\infty W^{1,\infty}$ ,

$$\mathbb{E} \int_0^t \int_D \gamma_\lambda(\nabla u_\lambda(s)) \cdot \nabla \varphi dx ds \longrightarrow \mathbb{E} \int_0^t \int_D \eta(s) \cdot \nabla \varphi dx ds,$$

hence  $\mathbb{E} \int_0^t \langle \operatorname{div} \gamma_\lambda(\nabla u_\lambda(s)), \varphi \rangle ds \longrightarrow \mathbb{E} \int_0^t \langle \operatorname{div} \eta(s), \varphi \rangle ds$ . Therefore, recalling (4), by difference we deduce that

$$\mathbb{E} \langle u_\lambda(t), \varphi \rangle \longrightarrow \mathbb{E} \langle u(t), \varphi \rangle.$$

Consequently, since  $u_\lambda(t)$  is bounded in  $L_\omega^2 L_x^2$ , we also have that  $u_\lambda(t) \rightarrow u(t)$  weakly in  $L_\omega^2 L_x^2$ . Taking the limit as  $\lambda \rightarrow 0$  in (4) thus yields

$$u(t) - \int_0^t \operatorname{div} \eta(s) ds = u_0 + \int_0^t G(s) dW(s) \quad \text{in } L_\omega^1 V_0',$$

where  $V_0'$  is the (topological) dual of a separable Hilbert space  $V_0$  embedded continuously and densely in  $H_0^1$ , and continuously in  $W^{1,\infty}$ . The identity immediately implies that  $u \in C_t L_\omega^1 V_0'$ . Since  $u \in L_t^\infty L_\omega^2 L_x^2$ , it follows by a result of Strauss (see [10, Theorem 2.1]) that  $u$  is a weakly continuous function on  $[0, T]$  with values in  $L_\omega^2 L_x^2$ .

By Mazur's lemma there exist sequences of convex combinations of  $\gamma_\lambda(\nabla u_\lambda)$  that converge  $\eta$  in (the norm topology of)  $L_{t,\omega,x}^1$ , thus also, passing to a subsequence,  $\mathbb{P} \otimes dt$ -almost everywhere in  $L_x^1$ . Similarly, since  $u_\lambda \rightarrow u$  weakly\* in  $L_t^\infty L_{\omega,x}^2$  implies that  $u_\lambda \rightarrow u$  weakly in  $L_{t,\omega,x}^2$ , there exist sequences of convex combinations of  $u_\lambda$  that converge to  $u$   $\mathbb{P} \otimes dt$ -almost everywhere in  $L_x^2$ . Since convex combinations of  $(u_\lambda)$  and of  $(\gamma_\lambda(\nabla u_\lambda))$  are (at least) predictable and adapted, respectively, it follows that  $u$  is predictable and  $\eta$  is measurable and adapted. Moreover, thanks to the weak lower semicontinuity of convex integrals, one has

$$\mathbb{E} \int_0^T \int_D (k(\nabla u) + k^*(\eta)) < \infty.$$

In order to show that  $\eta \in \gamma(\nabla u)$  for a.a.  $(\omega, t, x)$ , we need the following “energy identity”.

**Lemma 2.4.** *Assume that*

$$y(t) + \alpha \int_0^t y(s) ds - \int_0^t \operatorname{div} \zeta(s) ds = y_0 + \int_0^t C(s) dW(s)$$

in  $L_x^2$   $\mathbb{P}$ -a.s. for all  $t \in [0, T]$ , where  $\alpha \in \mathbb{R}$ ,  $y_0 \in L_{\omega, x}^2$  is  $\mathcal{F}_0$ -measurable, and

$$y \in L_t^\infty L_{\omega, x}^2 \cap L_{t, \omega}^1 W_0^{1,1}, \quad \zeta \in L_{t, \omega, x}^1, \quad C \in L_{t, \omega}^2 \mathcal{L}^2(H, L_x^2)$$

are measurable and adapted processes such that  $k(c\nabla y) + k^*(c\zeta) \in L_{t, \omega, x}^1$  for a constant  $c > 0$ . Then

$$\begin{aligned} \mathbb{E}\|y(t)\|^2 + 2\alpha \mathbb{E} \int_0^t \|y(s)\|^2 ds + 2 \mathbb{E} \int_0^t \int_D \zeta \cdot \nabla y dx ds \\ = \mathbb{E}\|y_0\|^2 + \mathbb{E} \int_0^t \|C(s)\|_{\mathcal{L}^2(H, L_x^2)}^2 ds \quad \forall t \in [0, T]. \end{aligned}$$

*Proof.* Let  $m \in \mathbb{N}$  be such that  $(I - \delta\Delta)^{-m}$  maps  $L_x^1$  into  $H_0^1 \cap W^{1, \infty}$ , and use the notation  $h^\delta := (I - \delta\Delta)^{-m} h$  for any  $h$  taking values in  $L_x^1$ . One has

$$y^\delta(t) + \alpha \int_0^t y^\delta(s) ds - \int_0^t \operatorname{div} \zeta^\delta(s) ds = y_0^\delta + \int_0^t C^\delta(s) dW(s) \quad (6)$$

$\mathbb{P}$ -a.s. for all  $t \in [0, T]$ , as an equality in  $L_x^2$ , for which Itô's formula yields

$$\begin{aligned} \|y^\delta(t)\|^2 + 2\alpha \int_0^t \|y^\delta(s)\|^2 ds + 2 \int_0^t \int_D \zeta^\delta \cdot \nabla y^\delta dx ds \\ = \|y_0^\delta\|^2 + \int_0^t \|C^\delta(s)\|_{\mathcal{L}^2(H, L_x^2)}^2 ds + \int_0^t y^\delta(s) C^\delta(s) dW(s). \end{aligned}$$

It is evident from (6) that  $y^\delta$  is a continuous  $L_x^2$ -valued process, hence the stochastic integral  $(y^\delta C^\delta) \cdot W$  on the right-hand side of the above identity is a continuous local martingale. Let  $(T_n)$  be a localizing sequence, and multiply the previous identity by  $1_{[0, T_n]}$ , to obtain, thanks to  $\mathbb{E}(y^\delta C^\delta) \cdot W(\cdot \wedge T_n) = 0$ ,

$$\begin{aligned} \mathbb{E}\|y^\delta(t \wedge T_n)\|^2 + 2\alpha \mathbb{E} \int_0^{t \wedge T_n} \|y^\delta(s)\|^2 ds + 2 \mathbb{E} \int_0^{t \wedge T_n} \int_D \zeta^\delta \cdot \nabla y^\delta dx ds \\ = \mathbb{E}\|y_0^\delta\|^2 + \mathbb{E} \int_0^{t \wedge T_n} \|C^\delta(s)\|_{\mathcal{L}^2(H, L_x^2)}^2 ds. \end{aligned}$$

Letting  $n$  tend to  $\infty$ , the dominated convergence theorem yields

$$\begin{aligned} \mathbb{E}\|y^\delta(t)\|^2 + 2\alpha \mathbb{E} \int_0^t \|y^\delta(s)\|^2 ds + 2 \mathbb{E} \int_0^t \int_D \zeta^\delta \cdot \nabla y^\delta dx ds \\ = \mathbb{E}\|y_0^\delta\|^2 + \mathbb{E} \int_0^t \|C^\delta(s)\|_{\mathcal{L}^2(H, L_x^2)}^2 ds \end{aligned}$$

for all  $t \in [0, T]$ . We are now going to pass to the limit as  $\delta \rightarrow 0$ : the first and second terms on the left-hand side and the first on the right-hand side clearly converge to  $\mathbb{E}\|y(t)\|^2$ ,  $2\alpha \mathbb{E} \int_0^t \|y(s)\|^2 ds$  and  $\mathbb{E}\|y_0\|^2$ , respectively. Properties of Hilbert-Schmidt operators and the dominated convergence theorem also yield

$$\lim_{\delta \rightarrow 0} \mathbb{E} \int_0^t \|C^\delta(s)\|_{\mathcal{L}^2(H, L_x^2)}^2 ds = \mathbb{E} \int_0^t \|C(s)\|_{\mathcal{L}^2(H, L_x^2)}^2 ds$$

for all  $t \in [0, T]$ . To conclude it then suffices to show that  $\nabla y^\delta \cdot \zeta^\delta \rightarrow \nabla y \cdot \zeta$  in  $L_{t, \omega, x}^1$ . Since  $\nabla y^\delta \rightarrow \nabla y$  and  $\zeta^\delta \rightarrow \zeta$  in measure in  $\Omega \times (0, t) \times D$ , Vitali's theorem implies strong convergence in  $L_{t, \omega, x}^1$  if the sequence  $(\nabla y^\delta \cdot \zeta^\delta)$  is uniformly integrable in  $\Omega \times (0, t) \times D$ . In turn, the latter is certainly true if  $(|\nabla y^\delta \cdot \zeta^\delta|)$  is dominated by a sequence that converges strongly in  $L_{t, \omega, x}^1$ . Indeed, using the assumptions on the behavior of  $k$  at infinity as well as the generalized Jensen inequality for sub-Markovian operators (see [2]), one has

$$\pm c^2 \zeta^\delta \cdot \nabla y^\delta \lesssim 1 + k(c \nabla y^\delta) + k^*(c \zeta^\delta) \leq 1 + (I - \delta \Delta)^{-m} (k(c \nabla y) + k^*(c \zeta)),$$

where the sequence on the right-hand side converges in  $L_{t, \omega, x}^1$  as  $\delta \rightarrow 0$  because, by assumption,  $k(c \nabla y) + k^*(c \zeta) \in L_{t, \omega, x}^1$ .  $\square$

Itô's formula yields

$$\begin{aligned} \mathbb{E}\|u_\lambda(t)\|^2 + 2 \mathbb{E} \int_0^t \int_D \gamma_\lambda(\nabla u_\lambda) \cdot \nabla u_\lambda + 2\lambda \mathbb{E} \int_0^t \|\nabla u_\lambda\|^2 \\ = \mathbb{E}\|u_0\|^2 + \mathbb{E} \int_0^t \|G(s)\|_{\mathcal{L}^2(H, L_x^2)}^2 ds \end{aligned}$$

and, by Lemma 2.4,

$$\mathbb{E}\|u(t)\|^2 + 2 \mathbb{E} \int_0^t \int_D \eta \cdot \nabla u = \mathbb{E}\|u_0\|^2 + \mathbb{E} \int_0^t \|G(s)\|_{\mathcal{L}^2(H, L_x^2)}^2 ds.$$



One then has

$$\begin{aligned}
& 2 \limsup_{\lambda \rightarrow 0} \mathbb{E} \int_0^T \int_D \gamma_\lambda(\nabla u_\lambda(s)) \cdot \nabla u_\lambda(s) dx ds \\
& \leq \mathbb{E} \|u_0\|^2 + \mathbb{E} \int_0^T \|G(s)\|_{\mathcal{L}^2(H, L_x^2)}^2 ds - \liminf_{\lambda \rightarrow 0} \mathbb{E} \|u_\lambda(T)\|^2 \\
& \leq \mathbb{E} \|u_0\|^2 + \mathbb{E} \int_0^T \|G(s)\|_{\mathcal{L}^2(H, L_x^2)}^2 ds - \mathbb{E} \|u(T)\|^2 \\
& = \mathbb{E} \int_0^T \int_D \eta(s) \cdot \nabla u(s) dx ds.
\end{aligned}$$

Since  $\nabla u_\lambda \rightarrow \nabla u$  and  $\gamma_\lambda(\nabla u_\lambda) \rightarrow \eta$  weakly in  $L_{t,\omega,x}^1$ , this implies that  $\eta \in \gamma(\nabla u)$  a.e. in  $\Omega \times (0, T) \times D$ . We have thus proved the existence and weak continuity statements of Proposition 2.1.

In order to show that the solution is unique, we are going to prove that *any* solution depends continuously on  $(u_0, G)$ . Let  $(u_i, \eta_i)$ ,  $i = 1, 2$ , satisfy

$$u_i(t) - \int_0^t \operatorname{div} \eta_i(s) ds = u_0 + \int_0^t G_i(s) ds$$

in the sense of Definition 1.1, as well as the integrability conditions (on  $u$  and  $\eta$ ) of Theorem 1.2. Setting  $y := u_1 - u_2$ ,  $y_0 := u_{01} - u_{02}$ ,  $\zeta := \eta_1 - \eta_2$ , and  $F := G_1 - G_2$ , one has

$$y(t) - \int_0^t \operatorname{div} \zeta(s) ds = y_0 + \int_0^t F(s) dW(s)$$

$\mathbb{P}$ -a.s. in  $L^2(D)$  for all  $t \in [0, T]$ . For any process  $h$ , let us use the notation  $h^\alpha(t) := e^{-\alpha t} h(t)$ . For any  $\alpha > 0$ , the integration-by-parts formula yields

$$y^\alpha(t) + \int_0^t (-\operatorname{div} \zeta^\alpha(s) + \alpha y^\alpha(s)) ds = y_0 + \int_0^t F^\alpha(s) dW(s),$$

hence also, thanks to Lemma 2.4,

$$\begin{aligned}
& \mathbb{E} \|y^\alpha(t)\|^2 + 2\alpha \mathbb{E} \int_0^t \|y^\alpha(s)\|^2 ds + 2 \mathbb{E} \int_0^t \int_D \zeta^\alpha(s) \cdot \nabla y^\alpha(s) dx ds \\
& \leq \mathbb{E} \|y_0\|^2 + \mathbb{E} \int_0^t \|F^\alpha(s)\|_{\mathcal{L}^2(H, L_x^2)}^2 ds,
\end{aligned}$$

where  $\zeta^\alpha \cdot \nabla y^\alpha \geq 0$  by monotonicity. Therefore, taking the  $L_t^\infty$  norm,

$$\|y^\alpha\|_{L_t^\infty L_{\omega,x}^2} + \sqrt{\alpha} \|y^\alpha\|_{L_{t,\omega,x}^2} \lesssim \|y_0\|_{L_{\omega,x}^2} + \|F^\alpha\|_{L_{t,\omega}^2 \mathcal{L}^2(H, L_x^2)},$$

that is, using the notation  $L_t^p(\alpha) := L^p([0, T], e^{-\alpha t} dt)$  for any  $p \geq 0$ ,

$$\begin{aligned}
& \|u_1 - u_2\|_{L_t^\infty(\alpha) L_{\omega,x}^2} + \sqrt{\alpha} \|u_1 - u_2\|_{L_t^2(\alpha) L_{\omega,x}^2} \\
& \lesssim \|u_{01} - u_{02}\|_{L_{\omega,x}^2} + \|G_1 - G_2\|_{L_t^2(\alpha) L_{\omega}^2 \mathcal{L}^2(H, L_x^2)}.
\end{aligned} \tag{7}$$

Taking  $\alpha = 0$  and  $G_1 = G_2$  immediately yields the uniqueness of solutions (as well as Lipschitz-continuous dependence on the initial datum). The proof of Proposition 2.1 is thus complete.

### 3 Proof of Theorem 1.2

For any  $v \in L^2_{t,\omega,x}$  measurable and adapted, and any  $\mathcal{F}_0$ -measurable random variable  $u_0 \in L^2_{\omega,x}$ , the process  $B(\cdot, v)$  is measurable, adapted, and belongs to  $L^2_{t,\omega}\mathcal{L}^2(H, L^2_x)$ , hence the equation

$$du(t) - \operatorname{div} \gamma(\nabla u(t)) dt \ni B(t, v(t)) dW(t), \quad u(0) = u_0,$$

is well-posed in the sense of Proposition 2.1. Moreover, for any  $v_1, v_2$  and  $u_{01}, u_{02}$  satisfying the same hypotheses on  $v$  and  $u_0$ , respectively, (7) yields

$$\begin{aligned} \|u_1 - u_2\|_{L^\infty_t(\alpha)L^2_{\omega,x}} + \sqrt{\alpha}\|u_1 - u_2\|_{L^2_t(\alpha)L^2_{\omega,x}} \\ \lesssim \|u_{01} - u_{02}\|_{L^2_{\omega,x}} + \|B(\cdot, v_1) - B(\cdot, v_2)\|_{L^2_t(\alpha)L^2_{\omega}\mathcal{L}^2(H, L^2_x)}. \end{aligned}$$

It hence follows by the Lipschitz-continuity of  $B$  that

$$\|u_1 - u_2\|_{L^2_t(\alpha)L^2_{\omega,x}} \lesssim \frac{1}{\sqrt{\alpha}} \left( \|u_{01} - u_{02}\|_{L^2_{\omega,x}} + \|v_1 - v_2\|_{L^2_t(\alpha)L^2_{\omega,x}} \right), \quad (8)$$

where the implicit constant does not depend on  $\alpha$ . In particular, denoting by  $\Gamma$  the map  $(u_0, v) \mapsto u$ , one has that  $v \mapsto \Gamma(u_0, v)$  is a strict contraction of  $L^2_t(\alpha)L^2_{\omega,x}$  for  $\alpha$  large enough. Therefore, by equivalence of norms,  $v \mapsto \Gamma(u_0, v)$  admits a unique fixed point in  $L^2_{t,\omega,x}$ , which solves (1) and satisfies all integrability conditions. Such solution is unique as any solution is a fixed point of  $v \mapsto \Gamma(u_0, v)$ .

Let us show that the solution map  $u_0 \mapsto u$  is Lipschitz-continuous: (8) yields, choosing  $\alpha$  large enough,

$$\|u_1 - u_2\|_{L^2_t(\alpha)L^2_{\omega,x}} \leq N_1 \|u_{01} - u_{02}\|_{L^2_{\omega,x}} + N_2 \|u_1 - u_2\|_{L^2_t(\alpha)L^2_{\omega,x}}$$

with constants  $N_1 > 0$  and  $0 < N_2 < 1$ , hence, by equivalence of norms,

$$\|u_1 - u_2\|_{L^2_t L^2_{\omega,x}} \lesssim \|u_{01} - u_{02}\|_{L^2_{\omega,x}}.$$

This in turn implies, in view of (7) (with  $\alpha = 0$ ) and the Lipschitz-continuity of  $B$ ,

$$\begin{aligned} \|u_1 - u_2\|_{L^\infty_t L^2_{\omega,x}} &\lesssim \|u_{01} - u_{02}\|_{L^2_{\omega,x}} + \|B(\cdot, u_1) - B(\cdot, u_2)\|_{L^2_{t,\omega}\mathcal{L}^2(H, L^2_x)} \\ &\lesssim \|u_{01} - u_{02}\|_{L^2_{\omega,x}} + \|u_1 - u_2\|_{L^2_t L^2_{\omega,x}} \lesssim \|u_{01} - u_{02}\|_{L^2_{\omega,x}}, \end{aligned}$$

which completes the proof.

*Remark.* A priori estimates entirely analogous to those of Lemma 2.2, as well as weak compactness results exactly as in Lemma 2.3, can be proved for the regularized equation obtained by replacing  $\gamma$  with  $\gamma_\lambda + \lambda \nabla$  directly in (1). It is however not immediately clear how to pass to the limit as  $\lambda \rightarrow 0$  in the stochastic integrals appearing in such regularized equations with multiplicative noise, i.e. to show that  $B(u_\lambda) \cdot W$  converges to  $B(u) \cdot W$  in a suitable sense.

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